[T.K.] 2.6.2.4 : Let $X \in N(0,1)$.

To show : $X \overset{d}{=} -X$, i.e $F_X(z) = F_{-X}(z)$ $\forall z \in \mathbb{R}$.

$$F_X(z) := \mathbb{P}(X \leq z) = \int_{(-\infty, z]} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

Now $F_{-X}(z) := \mathbb{P}(-X \leq z) = \mathbb{P}(X \geq -z) = \int_{[-z, +\infty)} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \underset{\tilde{x} = -x}{=} \int_{(-\infty, z]} \frac{1}{\sqrt{2\pi}} e^{-\tilde{x}^2/2} \, d\tilde{x} = \mathbb{P}(X \leq z) = F_X(z)$

Hence $X \overset{d}{=} -X$.

QED

[T.K.] 2.6.3.13 : let $X \in N(0,1)$ and $Y \in \chi^2(n)$ be two independent random variables.

To show: $\dfrac{X}{\sqrt{\frac{Y}{n}}} \in t(n)$.

Remmder : $X \in N(0,1) \iff f_X(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ $\forall t \in \mathbb{R}$.

$Y \in \chi^2(n) \iff f_Y(t) = \dfrac{t^{\frac{n}{2}-1} e^{-t/2}}{\Gamma(\frac{n}{2}) \, 2^{n/2}} \mathbb{1}_{\{t>0\}}(t)$ $\forall t \in \mathbb{R}$

$Z \in t(n) \iff f_Z(t) = \dfrac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi} \, \Gamma(\frac{n}{2})} \dfrac{1}{(1 + \frac{t^2}{n})^{\frac{n+1}{2}}}$ $\forall t \in \mathbb{R}$

First we need to understand what the p.d.f of a quotient of random variables is :

Let $X_1$ and $X_2$ be two independent random variables such that $X_2 > 0$. Fix $z \in \mathbb{R}$.

$$F_{\frac{X_1}{X_2}}(z) = \mathbb{P}\left(\frac{X_1}{X_2} \leq z\right) = \mathbb{P}(X_1 \leq z X_2, X_2 > 0) \underset{X_1, X_2 \text{ ind.}}{=} \int_0^{+\infty} \int_{-\infty}^{z x_2} f_{X_1}(x_1) f_{X_2}(x_2) \, dx_1 \, dx_2$$

Now $f_{\frac{X_1}{X_2}}(z) = \frac{d}{dz} F_{\frac{X_1}{X_2}}(z) = \frac{d}{dz} \int_0^{+\infty} \int_{-\infty}^{z x_2} f_{X_1}(x_1) f_{X_2}(x_2) \, dx_1 \, dx_2 = \int_0^{+\infty} \frac{d}{dz} \left\{ \int_{-\infty}^{z x_2} f_{X_1}(x_1) \, dx_1 \right\} f_{X_2}(x_2) \, dx_2$

$$= \int_0^{+\infty} f_{X_1}(z x_2) f_{X_2}(x_2) \cdot x_2 \, dx_2 . \qquad (1.)$$

Secondly we need to compute $f_{\sqrt{\frac{Y}{n}}}$ : $f_{\sqrt{\frac{Y}{n}}}(z) = \frac{d}{dz} \mathbb{P}\left(\sqrt{\frac{Y}{n}} \leq z\right) = \frac{d}{dz} \mathbb{P}\left(\frac{Y}{n} \leq z^2\right) = \frac{d}{dz} \mathbb{P}(Y \leq n z^2)$

$$= \frac{d}{dz} \int_0^{n z^2} f_Y(y) \, dy = \begin{cases} f_Y(n z^2) \cdot 2 n z & \forall z > 0 \\ 0 & \text{otherwise} \end{cases} \qquad (2.)$$

Combining these two steps we will now deduce the result :

$$f_{\frac{X}{\sqrt{\frac{Y}{n}}}}(z) \overset{(1)}{=} \int_0^{+\infty} f_X(zy)\, f_{\sqrt{\frac{Y}{n}}}(y)\, y\, dy \overset{(2)}{=} \int_0^{+\infty} f_X(zy)\, f_Y(ny^2) \cdot 2ny^2\, dy$$

$$= \int_0^{+\infty} \frac{n^{n/2} \cdot y^{n-1} \cdot e^{-ny^2/2}}{\Gamma(\frac{n}{2})\, 2^{\frac{n}{2}-1}}\, y\, \frac{1}{\sqrt{2\pi}}\, e^{-(zy)^2/2}\, dy = \frac{n^{n/2}}{\Gamma(\frac{n}{2})\, 2^{\frac{n}{2}-1}\sqrt{2\pi}} \underbrace{\int_0^{+\infty} e^{-\frac{1}{2}((zy)^2+ny^2)}\, y^n\, dy}_{} \qquad (*)$$

it has the shape of a Gamma-function !
Let's do a change of variables to write it properly.

$$\int_0^{+\infty} e^{-\frac{1}{2}((zy)^2+ny^2)} \cdot y^n\, dy = \int_0^{+\infty} e^{-1/2\, y^2(z^2+n)}\, y^n\, dy = \int_0^{+\infty} e^{-\tilde{y}} \cdot \left(\frac{2\tilde{y}}{(z^2+n)}\right)^{n/2} \frac{1}{\sqrt{2\tilde{y}}\sqrt{z^2+n}}\, d\tilde{y}$$

$$\tilde{y} = \frac{1}{2}y^2(z^2+n) \Leftrightarrow y = \sqrt{\frac{2\tilde{y}}{(z^2+n)}}$$

$$d\tilde{y} = (z^2+n)\, y\, dy = (z^2+n)\sqrt{\frac{2\tilde{y}}{(z^2+n)}}\, dy = \sqrt{2\tilde{y}}\sqrt{z^2+n}\, dy$$

$$= \left(\frac{2}{z^2+n}\right)^{n/2} \cdot \frac{1}{\sqrt{2}\sqrt{z^2+n}} \int_0^{+\infty} e^{-\tilde{y}} \cdot (\tilde{y})^{n/2} \cdot \frac{1}{\sqrt{\tilde{y}}}\, d\tilde{y} = \frac{2^{\frac{n-1}{2}}}{(z^2+n)^{\frac{n+1}{2}}} \underbrace{\int_0^{+\infty} e^{-\tilde{y}} \cdot \tilde{y}^{\frac{n-1}{2}}\, d\tilde{y}}_{\overset{def}{=}\Gamma(\frac{n+1}{2})}$$

Hence

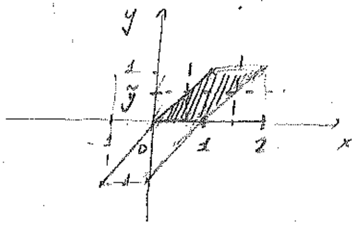
$$(*) = \frac{n^{n/2}}{\Gamma(\frac{n}{2})\, 2^{\frac{n}{2}-1}\sqrt{2\pi}} \cdot \frac{2^{\frac{n-1}{2}}}{(z^2+n)^{\frac{n+1}{2}}}\, \Gamma(\tfrac{n+1}{2}) = \frac{n^{n/2}\, \Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})\, (z^2+n)^{\frac{n+1}{2}}\sqrt{\pi}} = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})\left(1+\frac{z^2}{n}\right)^{\frac{n+1}{2}}\sqrt{n\pi}}$$

Hence $\frac{X}{\sqrt{\frac{Y}{n}}} \in t(n)$.

QED.

[T.K] 2.6.3.15 : If $(X,Y)$ has the probability distribution function :

$$f_{(X,Y)}(x,y) := \begin{cases} 1 & 0 \le x \le 2,\ \max(0,x-1) \le y \le \min(1,x) \\ 0 & \text{else where} \end{cases}$$



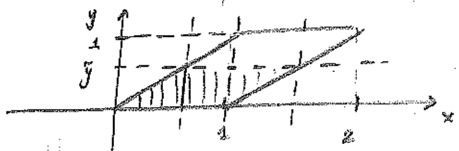
Then $X \in Tri(0,2)$ and $Y \in U(0,1)$

$$\mathbb{P}(X \le \tilde{x}) = \mathbb{P}((X,Y) ; X \le \tilde{x}, Y \in \mathbb{R}) = \int_0^{\tilde{x}} \left\{ \int_{\max(0,x-1)}^{\min(1,x)} 1 \, dy \right\} dx = \int_0^{\tilde{x}} (\min(1,x) - \max(0,x-1))\, dx .$$

Now $\min(1,x) - \max(0,x-1) = x \quad \forall x \in [0,1]$
$\min(1,x) - \max(0,x-1) = 2-x \quad \forall x \in [1,2]$

$$\Rightarrow f_X(x) = \begin{cases} x & \forall x \in [0,1] \\ 2-x & \forall x \in [1,2] \end{cases}$$

$\Rightarrow X \in Tri(0,2)$



Take $0 \le \tilde{y} \le 1$.

$$\mathbb{P}(Y \le \tilde{y}) = \mathbb{P}((X,Y) ; X \in [0,2], Y \le \tilde{y}) = \int_0^2 \int \mathbb{1}_{\{y \le \tilde{y}\}} \mathbb{1}_{\{\max(0,x-1) \le y \le \min(1,x)\}}\, dy\, dx$$

Note that $\max(0,x-1) \le y$ and $y \le \tilde{y}$ $\Rightarrow x-1 \le \tilde{y} \Leftrightarrow x \le 1+\tilde{y}$

$$= \int_0^1 \int \mathbb{1}_{\{y \le \tilde{y}\}} \mathbb{1}_{\{0 \le y \le \min(1,x)\}}\, dy\, dx + \int_1^{1+\tilde{y}} \int \underbrace{\mathbb{1}_{\{y \le \tilde{y}\}} \mathbb{1}_{\{x-1 \le y \le \min(1,x)\}}}\, dy\, dx$$

since $x$ ranges between 1 and 2, $\tilde{y} \le x$ and $\tilde{y} \le 1$
$\Rightarrow \mathbb{1}_{\{y \le \tilde{y}\}} \mathbb{1}_{\{x-1 \le y \le \min(1,x)\}} = \mathbb{1}_{\{x-1 \le y \le \tilde{y}\}}$.

$$= \int_0^{\tilde{y}} \int \underbrace{\mathbb{1}_{\{y \le \tilde{y}\}} \mathbb{1}_{\{0 \le y \le \min(1,x)\}}\, dy\, dx} + \int_{\tilde{y}}^1 \int \underbrace{\mathbb{1}_{\{y \le \tilde{y}\}} \mathbb{1}_{\{0 \le y \le \min(1,x)\}}\, dy\, dx} + \int_1^{1+\tilde{y}} \int \mathbb{1}_{\{x-1 \le y \le \tilde{y}\}}\, dy\, dx$$

$x$ ranges between 0 and $\tilde{y}$ $\Rightarrow x \le \tilde{y}$
$\Rightarrow \mathbb{1}_{\{y \le \tilde{y}\}} \mathbb{1}_{\{0 \le y \le \min(1,x)\}} = \mathbb{1}_{\{0 \le y \le x\}}$

$x$ ranges between $\tilde{y}$ and 1 $\Rightarrow x \ge \tilde{y}$
$\Rightarrow \mathbb{1}_{\{y \le \tilde{y}\}} \mathbb{1}_{\{0 \le y \le \min(1,x)\}} = \mathbb{1}_{\{0 \le y \le \tilde{y}\}}$

$$= \int_0^{\tilde{y}} \int_0^x dy\, dx + \int_{\tilde{y}}^1 \int_0^{\tilde{y}} dy\, dx + \int_1^{1+\tilde{y}} \int_{x-1}^{\tilde{y}} dy\, dx = \frac{1}{2}x^2 \Big|_0^{\tilde{y}} + \tilde{y}(1-\tilde{y}) + \int_1^{1+\tilde{y}} (\tilde{y} - (x-1))\, dx$$

$$= \frac{1}{2}\tilde{y}^2 + \tilde{y} - \tilde{y}^2 + \left(\tilde{y}x - \frac{1}{2}x^2 + x\right)\Big|_1^{1+\tilde{y}} = \frac{1}{2}\tilde{y}^2 + \tilde{y} - \tilde{y}^2 + \tilde{y}(1+\tilde{y}) - \frac{1}{2}(1+\tilde{y})^2 + (1+\tilde{y}) - \tilde{y} + \frac{1}{2} - 1 = \tilde{y}^2\left(\frac{1}{2} - 1 + 1 - \frac{1}{2}\right) + \tilde{y}(1+1-1+1-1) - \frac{1}{2} + 1 + \frac{1}{2} - 1$$

$= \tilde{y} \quad \Rightarrow Y \in U(0,1)$.

QED

[T.K] 2.6.3.17 : Suppose

$$f_{(X,Y)}(x,y) = \begin{cases} \dfrac{2}{(1+x+y)^3} & \forall x>0,\ y>0 \\ 0 & \text{elsewhere} \end{cases}$$

To show : (a) $f_{X+Y}(u) = \dfrac{2u}{(1+u)^3}$, $u > 0$

(b) $f_{X-Y}(v) = \dfrac{1}{2(1+|v|)^2}$ ; $v \in \mathbb{R}$

$\ast\ \mathbb{P}(X+Y\leq t)=\int_0^t f_{X+Y}(z)\,dz \quad \forall t\in\mathbb{R}$

Also $\mathbb{P}(X+Y\leq t)=\mathbb{P}((X,Y)\,;\,X\leq t-Y,\,Y\in\mathbb{R})=\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{t-y} f_{(X,Y)}(x,y)\,dx\right\}dy$

$\Rightarrow f_{X+Y}(t)=\frac{d}{dt}\mathbb{P}(X+Y\leq t)=\int_{-\infty}^{+\infty} f_{(X,Y)}(t-y,y)\,dy=\int_0^t \frac{2}{(1+t)^3}\,dy=\frac{2t}{(1+t)^3}$

$\ast\ \mathbb{P}(X-Y\leq t)=\int_0^t f_{X-Y}(z)dz \quad \forall t\in\mathbb{R}.$

and $\mathbb{P}(X-Y\leq t)=\mathbb{P}((X,Y)\,;\,X\leq t+Y,\,Y\in\mathbb{R})=\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{t+y} f_{(X,Y)}(x,y)\,dx\right\}dy$

$\Rightarrow f_{X-Y}(t)=\frac{d}{dt}\mathbb{P}(X-Y\leq t)=\int_{-\infty}^{+\infty} f_{(X,Y)}(t+y,y)\,dy \overset{t\,pos.}{=} \int_0^{+\infty}\frac{2}{(1+2y+t)^3}\,dy=-\frac{(1+2y+t)^{-2}}{2}\Big|_0^{+\infty}$

$=\frac{1}{2(1+t)^2}$

if $t$ is negative ; $\int_{-\infty}^{+\infty} f_{(X,Y)}(t+y,y)\,dy \underset{\substack{t+y>0\\ \Leftrightarrow\, y>-t}}{=} \int_{|t|}^{+\infty}\frac{2}{(1+2y+t)^3}\,dy=-\frac{(1+2y+t)^{-2}}{2}\Big|_{|t|}^{+\infty}=\frac{1}{2(1+|t|)^2}$

$\Rightarrow \forall t\in\mathbb{R},\ f_{X-Y}(t)=\underline{\underline{\frac{1}{2(1+|t|)^2}}}$

[T.K] 2.6.3.20 : Let $X_1 \in \Gamma(r,1)$ and $X_2 \in \Gamma(s,1)$ be independent.

To show : $\frac{X_1}{X_2}$ and $X_1+X_2$ are independent.

We want to see that $f_{(\frac{X_1}{X_2}, X_1+X_2)}(u,v) = f_{\frac{X_1}{X_2}}(u)\, f_{X_1+X_2}(v)$.

Very important Remark : It is enough to show that $f_{(\frac{X_1}{X_2}, X_1+X_2)}(u,v) = g_1(u)\cdot g_2(v)$ for some functions $g_1, g_2$.

So we just have to compute $f_{(\frac{X_1}{X_2}, X_1+X_2)}(u,v)$.

Fix $t_1, t_2 \in \mathbb{R}$. $\mathbb{P}\left(\left(\frac{X_1}{X_2}, X_1+X_2\right) \in (-\infty,t_1]\times(-\infty,t_2]\right) = \mathbb{P}\left((X_1,X_2) \in F\left((-\infty,t_1]\times(-\infty,t_2]\right)\right)$

We want to find $F$ such that

$F\left(\frac{X_1}{X_2}, X_1+X_2\right) = (X_1, X_2)$ Set $U = \frac{X_1}{X_2}$ and $V = X_1+X_2$

$$\begin{cases} U = \frac{X_1}{X_2} \\ V = X_1+X_2 \end{cases} \Leftrightarrow \begin{cases} U = \frac{V-X_2}{X_2} \\ X_1 = V - X_2 \end{cases} \Leftrightarrow \begin{cases} X_2 U + X_2 = V \\ X_1 = V - X_2 \end{cases}$$

$$\Leftrightarrow \begin{cases} X_2 = \frac{V}{1+U} \\ X_1 = V - \frac{V}{1+U} = \frac{V+UV-V}{1+U} = \frac{UV}{1+U} \end{cases}$$

$$= \iint_{F((-\infty,t_1]\times(-\infty,t_2])} f_{(X_1,X_2)}(x_1,x_2)\,dx_2\,dx_1 \overset{X_1 \text{ and } X_2 \text{ ind.}}{=} \iint_{F((-\infty,t_1]\times(-\infty,t_2])} f_{X_1}(x_1) f_{X_2}(x_2)\,dx_2\,dx_1 = \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} f_{X_1}\left(\frac{uv}{1+u}\right) f_{X_2}\left(\frac{v}{1+u}\right)\frac{v}{(1+u)^2}\,dv\,du$$

change of variables :

$(x_1,x_2)$ to $(U,V)$

$$J_F = \begin{pmatrix} \frac{dx_1}{dU} & \frac{dx_1}{dV} \\ \frac{dx_2}{dU} & \frac{dx_2}{dV} \end{pmatrix} = \begin{pmatrix} \frac{v(1+u)-uv}{(1+u)^2} & \frac{u}{1+u} \\ \frac{-v}{(1+u)^2} & \frac{1}{1+u} \end{pmatrix}$$

$$\Rightarrow |\det J_F| = \frac{v}{(1+u)^2}$$

$$= \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} \frac{1}{\Gamma(r)\Gamma(s)} e^{-\left(\frac{uv}{1+u}+\frac{v}{1+u}\right)} \left(\frac{uv}{1+u}\right)^{r-1}\left(\frac{v}{1+u}\right)^{s-1}\frac{v}{(1+u)^2}\,dv\,du$$

$$= \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} \frac{1}{\Gamma(r)\Gamma(s)} e^{-v} \frac{1}{(1+u)^{r+s}} u^{r-1} v^{r+s-1}\,dv\,du$$

Now since $\int_{-\infty}^{t_1}\int_{-\infty}^{t_2} f(u,v)\,du\,dv = \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} g(u,v)\,du\,dv \;\; \forall t_1,t_2 \in \mathbb{R} \Rightarrow f = g$ almost surely

and $\mathbb{P}\left(\left(\frac{X_1}{X_2}, X_1+X_2\right) \in (-\infty,t_1]\times(-\infty,t_2]\right) = \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} f_{(\frac{X_1}{X_2}, X_1+X_2)}(u,v)\,dv\,du$

and $\mathbb{P}\left(\left(\frac{X_1}{X_2}, X_1+X_2\right) \in (-\infty,t_1]\times(-\infty,t_2]\right) = \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} \frac{1}{\Gamma(r)\Gamma(s)} e^{-v}\frac{1}{(1+u)^{r+s}} u^{r-1} v^{r+s-1}\,dv\,du$

$\Rightarrow f_{(\frac{X_1}{X_2}, X_1+X_2)}(u,v) = \frac{1}{\Gamma(r)\Gamma(s)} e^{-v}\frac{1}{(1+u)^{r+s}} u^{r-1} v^{r+s-1}$ and $\frac{X_1}{X_2}$ and $X_1+X_2$ are independent !

QED

MOREILLON Philippe SF2940 (Additional Exercise)

[T.K] 2.6.8.5 Let $X \geq 0$ and $Y \geq 0$ be independent continuous random variables with $f_X(x)$ and $f_Y(y)$ their respective densities. Show that $XY$ has p.d.f :

$$f_{XY}(x) = \int_0^{+\infty} \frac{1}{y} f_X\left(\frac{x}{y}\right) f_Y(y)\, dy = \int_0^{+\infty} \frac{1}{y} f_X(y) f_Y\left(\frac{x}{y}\right) dy$$

$$\mathbb{P}(XY \leq x) \underset{\substack{/ \\ \mathbb{P}(Y=0)=0 \text{ since } Y \text{ is continuous}}}{=} \mathbb{P}\left(X \leq \frac{x}{Y}, Y > 0\right) \overset{\text{ind}}{=} \int_0^{+\infty} \int_0^{x/v} f_X(u) f_Y(v)\, du\, dv$$

Set $t \mapsto F_1(t) = \int_0^t f_X(u) f_Y(v)\, du$ Then $\mathbb{P}(XY \leq x) = \int_0^{+\infty} F_1\left(\frac{x}{v}\right) dv$. Also by definition $\mathbb{P}(XY \leq x) = \int_0^x f_{XY}(u)\, du$

$t \mapsto F_2(t) = \int_0^t f_{XY}(u)\, du$ $= F_2(x)$.

By the Fundamental Theorem of analysis, $\frac{d}{dt} F_1(t) = f_Y(v) \cdot f_X(t)$ and $\frac{d}{dt} F_2(t) = f_{XY}(t)$

$$\Rightarrow \frac{d}{dx} \mathbb{P}(XY \leq x) = \frac{d}{dx} F_2(x) = f_{XY}(x)$$

$$\text{and } \frac{d}{dx} \mathbb{P}(XY \leq x) = \frac{d}{dx} \int_0^{+\infty} F_1\left(\frac{x}{v}\right) dv \underset{\substack{/ \\ \text{the densities are continuous}}}{=} \int_0^{+\infty} \left(\frac{d}{dx} F_1\left(\frac{x}{v}\right)\right) dv \underset{/}{=} \int_0^{+\infty} \frac{1}{v} f_X\left(\frac{x}{v}\right) f_Y(v)\, dv$$

$$\frac{d}{dx} F_1\left(\frac{x}{v}\right) = \underbrace{\frac{d}{dx}\Big|_{\frac{x}{v}} F_1}_{= f_Y(v) \cdot f_X\left(\frac{x}{v}\right)} \cdot \underbrace{\frac{d}{dx} \frac{x}{v}}_{= \frac{1}{v}} = \frac{1}{v} f_X\left(\frac{x}{v}\right) f_Y(v)$$

$$\Rightarrow f_{XY}(x) = \int_0^{+\infty} \frac{1}{v} f_X\left(\frac{x}{v}\right) f_Y(v)\, dv.$$

$$\underset{/}{=} \int_0^{+\infty} \frac{1}{v} f_X(v) f_Y\left(\frac{x}{v}\right) dv$$

By symmetry (just start by $\mathbb{P}(XY \leq x) = \mathbb{P}\left(Y \leq \frac{x}{X}, X > 0\right)$ instead)

QED.

<u>Technical Drill</u> : $X \in U(0,1)$ and $Y \in U(0,1)$ independent and $W := XY$. Show that $f_W(w) = -\ln w$, $0 < w \leq 1$.

Take $x$ positive.

$$f_W(w) = f_{XY}(w) = \int_0^{+\infty} \frac{1}{v} f_X\left(\frac{w}{v}\right) f_Y(v)\, dv \underset{/}{=} \int_0^{+\infty} \frac{1}{v} \cdot \mathbb{1}_{[0,1]}\left(\frac{w}{v}\right) \mathbb{1}_{[0,1]}(v)\, dv \underset{/}{=} \int_w^1 \frac{1}{v}\, dv = \ln(v)\Big|_w^1 = -\ln(w), \quad 0 < w \leq 1.$$

$\mathbb{P}(Y \leq y) = y \Leftrightarrow f_Y(y) = \mathbb{1}_{[0,1]}(y)$ $\frac{w}{v} \leq 1 \Leftrightarrow w \leq v \rightarrow w \in [0,1]$

$\mathbb{P}(X \leq x) = x \Leftrightarrow f_X(x) = \mathbb{1}_{[0,1]}(x)$